ON BOUNDED PEAKING IN THE CHEAP CONTROL REGULATOR

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Abstract. The cheap control regulator for time-invariant nonlinear systems is studied with respect to uniform L^2-boundedness of the state trajectories in the case where the small parameter "ϵ" tends to zero. By using the geometric approach, state-space conditions for L^2-boundedness are found and related to the concept of zero dynamics

Key Words. Optimal control, stabilization, singular perturbation, time invariant systems, transmission zeroes

1 INTRODUCTION

In many control systems, a strong control action is desirable. For feedback systems, a strong control action is achieved by using high gain controls. In order to obtain a high gain regulator as the result of an optimal control problem, the cost of the control should be "cheap". For a linear control system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx}, \\ \mathbf{x} &\in R^n,\ \mathbf{y} \in R^p,\ \mathbf{u} \in R^m, \end{aligned} \qquad (1)$$

the cheap control problem is characterized by the presence of the small positive constant ϵ in the cost functional

$$J = \frac{1}{2}\int_0^\infty (\|\mathbf{Cx}\|^2 + \epsilon^2\|\mathbf{u}\|^2)dt. \qquad (2)$$

The cheap control problem has been widely studied in the literature, see for example (O'Malley and Jameson, 1975; Young *et al.*, 1977; Kokotovic, 1984) and the references therein.

When $\mathbf{u}(\cdot)$ minimizes (2), the resulting state trajectory $\phi_\epsilon(\mathbf{x}_0)$ of (1) obviously depends on ϵ. Francis and Glover (1978) studied the following important problem: when is the trajectory $\phi_\epsilon(\mathbf{x}_0)$ bounded uniformly as ϵ tends to zero, for each $\mathbf{x}_0$? This is actually what the bounded peaking means. Naturally, this problem is of considerable practical interest. For example, in many cases, one would like to increase the speed of response of the system by tuning the gain as high as possible, then he must be first assured that this can be done without causing some state variables to peak excessively.

As is well known, the optimal control of (2) is a linear feedback control $\mathbf{u} = \mathbf{F}_\epsilon\mathbf{x}$ where

$$\mathbf{F}_\epsilon = -\frac{1}{\epsilon^2}\mathbf{B}^T\mathbf{P}_\epsilon$$

and $\mathbf{P}_\epsilon$ is the positive semidefinite solution of the Riccati equation

$$\mathbf{A}^T\mathbf{P}_\epsilon + \mathbf{P}_\epsilon\mathbf{A} + \mathbf{C}^T\mathbf{C} = \frac{1}{\epsilon^2}\mathbf{P}_\epsilon\mathbf{B}\mathbf{B}^T\mathbf{P}_\epsilon. \qquad (3)$$

The closed-loop system is described by

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}\mathbf{F}_\epsilon)\mathbf{x} \tag{4}$$

The transition matrix of (4) is

$$\mathbf{T}_\epsilon = e^{(\mathbf{A}+\mathbf{B}\mathbf{F}_\epsilon)t}.$$

By L^2-bounded peaking here means that the set

$$\{\|\mathbf{T}_\epsilon\|_2 : \ \epsilon_0 \geq \epsilon > 0\}$$

is bounded for some $\epsilon_0 > 0$.

In (Francis and Glover, 1978) the following necessary and sufficient conditions are given:

Lemma 1.1 L^2-bounded peaking is equivalent to the conditions

1. rank $\mathbf{G}(s)$ =rank $\mathbf{CB} = r$, and
2. $\gamma_r(s^2)$ has no zeros in $Re\ s = 0$,

where $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ and $\gamma_r = det(s\mathbf{I} - \mathbf{A})det(-s\mathbf{I} - \mathbf{A})\bar{\gamma}_r$, where $\bar{\gamma}_r$ is the sum of all r-order principal minors of the matrix $\mathbf{G}(s)\mathbf{G}(-s)^T$.

When these conditions are satisfied, the solution to the cheap control problem always results in a high gain (consider $k = \frac{1}{\epsilon}$ as the gain) feedback law. As ϵ tends to zero, the trajectories of the closed-loop system converge to a "slow" invariant subspace. In this paper is given a complete characterization of the dynamics on this subspace from a geometric point of view, and some of the results are generalized to nonlinear affine control systems. It should be pointed out that for linear systems, by using a special coordinate system, the converging subspace was identified in (Saberi and Sannuti, 1987). However, the approach used here is coordinate free and it will be seen that computation of the dynamics on the subspace is actually quite subtle in the case where the system does not have a relative degree.

2 PRELIMINARIES

In this section is given a brief review of the concepts of zero dynamics and adjoint system which will be needed later. Consider a linear system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}, \end{aligned} \tag{5}$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$ and $\mathbf{y} \in R^p$.

For (5), the adjoint system is defined as:

$$\begin{aligned} \dot{\mathbf{z}} &= -\mathbf{A}^T\mathbf{z} + \mathbf{C}^T\mathbf{v} \\ \mathbf{w} &= \mathbf{B}^T\mathbf{z}. \end{aligned} \tag{6}$$

Using $\mathbf{G}(s)$ and $\mathbf{G}_a(s)$ to denote the transfer function matrices for (5) and (6) respectively, it is well known that the following equality holds:

$$\mathbf{G}(s) = -\mathbf{G}_a^*(-s).$$

In particular one sees that the transmission zeros of the adjoint system are the mirror images of those of the original system. The concept of zero dynamics, see for example (Isidori, 1989), is closely associated with transmission zeros and has proven to be quite useful, especially for nonlinear systems, where transmission zeros are not definable.

In this paper the adjoint zero dynamics of the zero dynamics of (5) is defined as the zero dynamics of (6).

Now consider a nonlinear system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}), \end{aligned} \tag{7}$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$ and $\mathbf{y} \in R^p$, and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, $\mathbf{h}(\mathbf{0}) = \mathbf{0}$, and $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), ..., g_m(\mathbf{x}))$. For the sake of simplicity, it is also assumed that all the mappings are smooth in a neighborhood $N(\mathbf{0})$ of $\mathbf{0}$.

For any $\mathbf{x}_0$ in $N(\mathbf{0})$, the pointwise adjoint system of (7) is defined as

$$\begin{aligned} \dot{\mathbf{z}} &= -(\tfrac{\partial \mathbf{f}(\mathbf{x}_0)}{\partial \mathbf{x}})^T\mathbf{z} + (\tfrac{\partial \mathbf{h}(\mathbf{x}_0)}{\partial \mathbf{x}})^T\mathbf{v} \\ \mathbf{w} &= (\mathbf{g}(\mathbf{x}_0))^T\mathbf{z}. \end{aligned} \tag{8}$$

Similarly, the pointwise adjoint zero dynamics of the zero dynamics of (7) is defined as the zero dynamics of (8). Obviously, for nonlinear systems the pointwise adjoint zero dynamics makes sense only when the zero dynamics of the original system does exist.

3 BOUNDED PEAKING

Consider the optimal control problem

$$\min \int_0^\infty (\mathbf{h}(\mathbf{x})^T\mathbf{h}(\mathbf{x}) + \epsilon^2\mathbf{u}^T\mathbf{u})dt$$

subject to the system (7) under the following assumption.

A1: For any $\epsilon > 0$, the above optimal control problem is solvable for any $\mathbf{x}$ in a neighborhood $W(\mathbf{0})$ of $\mathbf{0}$ and the optimal control is in the form of a smooth feedback $\mathbf{u} = \mathbf{u}_\epsilon(\mathbf{x})$.

First a special case of (7) is considered, namely where $p = m$, i.e., the number of inputs is equal to the number of outputs. In this case a second assumption is made:

A2: The system (7) has relative degree $(1, ..., 1)$ at $\mathbf{x} = \mathbf{0}$ and $\text{span}\{g_1(\mathbf{x}), ..., g_m(\mathbf{x})\}$ is involutive.

Remark: As long as the system has a relative degree, it is easy to see that the relative degree being one is a necessary condition for the closed-loop trajectories to be L^2-bounded.

Under the hypothesis A1, it is a standard result that for (7) both the zero dynamics and its pointwise adjoint exist at least locally. Let V^* denote the zeroing subspace associated with the pointwise adjoint zero dynamics at a point $\mathbf{x} \in W(\mathbf{0})$ and $\mathbf{F}(\mathbf{x})$ be a friend of V^* (see (Wonham, 1979) for the definition of friend). Now let a new output to the original system be defined as

$$\tilde{\mathbf{y}} = \mathbf{y} + \mathbf{F}(\mathbf{x})\mathbf{z}.$$

It turns out that, as $\epsilon \to 0$, the closed-loop trajectories of the optimal system actually converges to the zero dynamics of the following augmented system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \\ \dot{\mathbf{z}} &= -(\tfrac{\partial \mathbf{f}(\mathbf{x}_0)}{\partial \mathbf{x}})^T\mathbf{z} + (\tfrac{\partial \mathbf{h}(\mathbf{x}_0)}{\partial \mathbf{x}})^T\mathbf{v} \\ \tilde{\mathbf{y}} &= \mathbf{h}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\mathbf{z} \\ \mathbf{w} &= (\mathbf{g}(\mathbf{x}_0))^T\mathbf{z}, \end{aligned} \tag{9}$$

where $\tilde{\mathbf{y}}$ and $\mathbf{w}$ are considered as the output. This is summarized below.

Theorem 3.1 Suppose the hypotheses A1 and A2 are satisfied and the zero dynamics of (7) is hyperbolic. Then, there exists a neighborhood $W'(\mathbf{0}) \subset W(\mathbf{0})$, such that for all initial conditions in $W'(\mathbf{0})$, as $\epsilon \to 0$, the closed-loop trajectories of the optimal system converges to the zero dynamics of the augmented system (9) defined on M^*. Furthermore, there is a stable manifold $Z^* = \{(\mathbf{x}, \mathbf{z}) \in M^* : \mathbf{z} = \phi(\mathbf{x})\}$, such that the closed-loop trajectories of the optimal system converges to Z^*.

The proof is just a fairly standard application of singular perturbation methods. It is omitted here.

Under the hypothesis A2, one can transform (7) into the following normal form locally:

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}, \eta) \\ \dot{\eta} &= \mathbf{f}_1(\mathbf{z}, \eta) + \mathbf{g}_1(\mathbf{z}, \eta)\mathbf{u} \\ \mathbf{y} &= \eta, \end{aligned} \tag{10}$$

where $\mathbf{g}_1(\mathbf{0}, \mathbf{0})$ is nonsingular. One can easily compute the zero dynamics of the augmented system as

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}, \mathbf{p}(\mathbf{z}, \mathbf{z}^*)) \\ \dot{\mathbf{z}}^* &= -(\tfrac{\partial \mathbf{f}_0(\mathbf{z}, \mathbf{p}(\mathbf{z}, \mathbf{z}^*))}{\partial \mathbf{z}})^T\mathbf{z}^*, \end{aligned} \tag{11}$$

where $\mathbf{p}(\mathbf{z}, \mathbf{z}^*)$ is implicitly defined by

$$\mathbf{p} + (\frac{\partial \mathbf{f}_0(\mathbf{z}, \mathbf{p})}{\partial \mathbf{y}})^T\mathbf{z}^* = \mathbf{0}.$$

In the linear case, (11) becomes

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}_0\mathbf{z} - \mathbf{A}_1\mathbf{A}_1^T\mathbf{z}^* \\ \dot{\mathbf{z}}^* &= -\mathbf{A}_0^T\mathbf{z}^* \end{aligned} \tag{12}$$

where the pair $(\mathbf{A}_0, \mathbf{A}_1)$ is stabilizable by A1. Thus, when the zero dynamics is stable, the dynamics of the converging optimal trajectories is governed by the zero dynamics

$$\dot{\mathbf{z}} = \mathbf{A}_0\mathbf{z};$$

when the zero dynamics is antistable, the dynamics of the converging optimal trajectories is governed by

$$\dot{\mathbf{z}}^* = -\mathbf{P}\mathbf{A}_0^T\mathbf{z}\mathbf{P}^{-1},$$

where $\mathbf{P}$ is the solution to a Lyapunov equation. When $\mathbf{A}_0$ contains both stable and unstable eigenvalues, one has to decompose $\mathbf{A}_0$ further.

The general case, where the system is not necessarily square and/or does not necessarily have relative degree 1, is quite difficult to analyze. Here only the linear case is discussed, as the preparation for eventually solving the nonlinear case.

In the linear case, the optimal control problem becomes

$$\min \frac{1}{2}\int_0^\infty (||\mathbf{y}||^2 + \epsilon^2||\mathbf{u}||^2)dt \tag{13}$$

subject to

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}, \end{aligned} \tag{14}$$

under the assumptions that $(\mathbf{A}, \mathbf{B})$ is stabilizable, $(\mathbf{C}, \mathbf{A})$ is detectable, $\mathbf{B}$ has linearly independent columns and $\mathbf{C}$ has linearly independent rows. Now a coordinate free characterization of the L^2-bounded peaking conditions is discussed and from the proof the dynamical equations which govern the converging optimal trajectories, as ϵ tends to 0, are given.

Theorem 3.2 L^2-bounded peaking is equivalent to the conditions that
1. the restriction of $\mathbf{C}$ to the subspace $B/(B \cap R^*)$ of the state-space has full column rank, where $B = \text{Im}(\mathbf{B})$, and
2. the zero dynamics of the system is hyperbolic.

Considering the decomposition suggested by Francis and Glover (1978), the state-space X can be decomposed as

$$\begin{aligned} X &= X_1 \oplus X_2 \oplus X_3 \oplus X_4 \\ &= V^*/R^* \oplus R^* \oplus W \oplus B/(B \cap R^*), \end{aligned}$$

where W is the complement of $V^* \oplus B$ in X. The system (14) then has the following form

$$\begin{aligned} \begin{pmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_3 \\ \dot{\mathbf{x}}_4 \end{pmatrix} &= \begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{13} & \mathbf{A}_{14} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{A}_{24} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{A}_{41} & \mathbf{A}_{42} & \mathbf{A}_{43} & \mathbf{A}_{44} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix} \\ &+ \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_4 \end{pmatrix} \begin{pmatrix} \mathbf{u}_2 \\ \mathbf{u}_4 \end{pmatrix}, \\ \mathbf{y} &= (\mathbf{0} \;\; \mathbf{0} \;\; \mathbf{C}_3 \;\; \mathbf{C}_4)\mathbf{x}, \end{aligned}$$

where $\mathbf{B}_4$ is nonsingular. Following the standard procedure to solve the problem (13)- (14), the variables λ_2, λ_4 and $\mathbf{x}_2$ are rescaled according to $\bar{\lambda}_2 := \lambda_2/\epsilon$, $\bar{\lambda}_4 := \lambda_4/\epsilon$ and $\hat{\mathbf{x}}_2 := \epsilon\mathbf{x}_2$. The new variables converge according to Francis and Glover (1978). In order to have the singularly perturbated subsystem corresponding to the variables $\mathbf{x}_4$ and λ_4 in standard form, the matrix $\mathbf{C}_4^T\mathbf{C}_4$ must be nonsingular, i.e., $\mathbf{C}_4$ must have full column rank, which is assumed in the sequel. First consider the case where $\mathbf{C}_3^T\mathbf{C}_4 = \mathbf{0}$ and let $\mathbf{C}_4^T\mathbf{C}_4 = \mathbf{I}$ to simplify the notation. The reduced dynamic equations, as $\epsilon - 0$, then become

$$\begin{aligned} \dot{\hat{\mathbf{x}}}_2 &= \mathbf{A}_{22}\hat{\mathbf{x}}_2 - \mathbf{B}_2\mathbf{B}_2^T\bar{\lambda}_2 \\ \dot{\bar{\lambda}}_2 &= -\mathbf{A}_{22}^T\bar{\lambda}_2 - \mathbf{A}_{42}^T(\mathbf{B}_4\mathbf{B}_4^T)^{-1}\mathbf{A}_{42}\hat{\mathbf{x}}_2 \end{aligned} \tag{15}$$

and

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{A}_{13}\mathbf{x}_3 - \mathbf{A}_{14}\mathbf{A}_{14}^T\lambda_1 \\ &\quad -\mathbf{A}_{14}\mathbf{A}_{34}^T\lambda_3 \\ \dot{\lambda}_1 &= -\mathbf{A}_{11}^T\lambda_1 \\ \dot{\mathbf{x}}_3 &= \mathbf{A}_{33}\mathbf{x}_3 - \mathbf{A}_{34}\mathbf{A}_{14}^T\lambda_1 - \mathbf{A}_{34}\mathbf{A}_{34}^T\lambda_3 \\ \dot{\lambda}_3 &= -\mathbf{C}_3^T\mathbf{C}_3\mathbf{x}_3 - \mathbf{A}_{13}^T\lambda_1 - \mathbf{A}_{33}^T\lambda_3. \end{aligned} \tag{16}$$

The dynamic equations corresponding to the subspaces R^* and $W \cup V^*/R^*$ are two independent subsystems. First, consider the subspace R^*. In the equations (15), the pair $(\mathbf{A}_{22}, \mathbf{B}_2)$ is of course stabilizable. Furthermore, $(\mathbf{A}_{42}, \mathbf{A}_{22})$ must be detectable from the assumption of overall detectability of the system. Therefore the Riccati equation associated with (15) has a unique positive semidefinite stabilizing solution $\hat{\mathbf{P}}_2$. Now suppose that $\mathbf{P}$ is the possible solution to the Riccati equation associated with (16). Partition $\mathbf{P}$ as

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{P}_2^T & \mathbf{P}_4 \end{pmatrix}$$

Let $\mathbf{P}_4 = \bar{\mathbf{P}}_4 + \hat{\mathbf{P}}_4$, where $\hat{\mathbf{P}}_4$ is the unique positive semidefinite solution to the equation

$$\mathbf{A}_{33}^T\hat{\mathbf{P}}_4 + \hat{\mathbf{P}}_4\mathbf{A}_{33} - \hat{\mathbf{P}}_4\mathbf{A}_{34}\mathbf{A}_{34}^T\hat{\mathbf{P}}_4 + \mathbf{C}_3^T\mathbf{C}_3 = \mathbf{0}.$$

Such a solution $\hat{\mathbf{P}}_4$ exists, since clearly $(\mathbf{A}_{33}, \mathbf{A}_{34})$ is stabilizable and $(\mathbf{C}_3, \mathbf{A}_{33})$ is observable.

The Riccati equation can therefore be written

$$\mathbf{P}\bar{\mathbf{A}} + \bar{\mathbf{A}}^T\mathbf{P} - \mathbf{P}\bar{\mathbf{B}}\bar{\mathbf{B}}^T\mathbf{P} = 0,$$

where

$$\bar{\mathbf{A}} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{13} - \mathbf{A}_{14}\mathbf{A}_{34}^T\hat{\mathbf{P}}_4 \\ \mathbf{0} & \mathbf{A}_{33} - \mathbf{A}_{34}\mathbf{A}_{34}^T\hat{\mathbf{P}}_4 \end{pmatrix}, \bar{\mathbf{B}} = \begin{pmatrix} \mathbf{A}_{14} \\ \mathbf{A}_{34} \end{pmatrix}$$

It is immediately seen that since $\mathbf{A}_{33} - \mathbf{A}_{34}\mathbf{A}_{34}^T\hat{\mathbf{P}}_4$ is stable, a unique positive semidefinite solution exists if $\mathbf{A}_{11}$ does not have any eigenvalues on the imaginary axis.

Now consider a general matrix $\mathbf{C}$, which has linearly independent rows. A straightforward calculation shows that the form of the Riccati equation for the case $\mathbf{C}_3^T\mathbf{C}_4 = 0$ is preserved by substituting $\mathbf{A}_{13}$, $\mathbf{A}_{33}$ and $\mathbf{C}_3^T\mathbf{C}_3$ for

$$\begin{aligned}\bar{\mathbf{A}}_{13} &:= \mathbf{A}_{13} - \mathbf{A}_{14}(\mathbf{C}_4^T\mathbf{C}_4)^{-1}\mathbf{C}_4^T\mathbf{C}_3 \\ \bar{\mathbf{A}}_{33} &:= \mathbf{A}_{33} - \mathbf{A}_{34}(\mathbf{C}_4^T\mathbf{C}_4)^{-1}\mathbf{C}_4^T\mathbf{C}_3 \\ \bar{\mathbf{C}}_3^T\bar{\mathbf{C}}_3 &:= \mathbf{C}_3^T(\mathbf{I} - \mathbf{C}_4(\mathbf{C}_4^T\mathbf{C}_4)^{-1}\mathbf{C}_4^T)\mathbf{C}_3,\end{aligned}$$

respectively. It is easy to show that $(\bar{\mathbf{A}}_{33}, \mathbf{A}_{34})$ is stabilizable. Defining the orthogonal projection matrix $\mathbf{P}_o := \mathbf{C}_4(\mathbf{C}_4^T\mathbf{C}_4)^{-1}\mathbf{C}_4^T$, $\mathbf{P}_o\mathbf{C}_3$ can be interpreted as the orthogonal projection of $\mathbf{C}_3$ onto the column space of $\mathbf{C}_4$. The following lemma reflects the subtlety in the general case.

Lemma 3.3 The pair $(\bar{\mathbf{C}}_3, \bar{\mathbf{A}}_{33})$ is observable.

Proof: Denote by a bar matrices and subspaces referring to the system "after projection", i.e., after having made the substitutions above, and let matrices and subspaces without bar refer to the original decomposed system. Suppose that $(\bar{\mathbf{C}}_3, \bar{\mathbf{A}}_{33})$ is not observable. Then $\bar{V}^*$ is not maximal. Furthermore, $\mathbf{C}_3$ and $\mathbf{C}_4$ cannot form set of linearly independent column vectors. In the following is shown that there exists a subspace $V_3 \neq 0$, such that $V := V^* + V_3$ contradicts the maximality of V^*, exactly when there exists a $\bar{V}_3 \neq 0$ that contradicts the maximality of $\bar{V}^*$. Note that $\bar{V}^* \neq V^*$ in general. Since $\ker[\mathbf{C}_3, \mathbf{C}_4] \neq 0$, there are subspaces X_3 and X_4 such that $\mathbf{C}_3X_3 + \mathbf{C}_4X_4 = 0$. Multiplying by $\mathbf{C}_4^T$ and inverting, yields $X_4 = -(\mathbf{C}_4^T\mathbf{C}_4)^{-1}\mathbf{C}_4^T\mathbf{C}_3X_3$. Let

$$V_3 = (\,0 \quad 0 \quad \bar{X}_3 \quad \bar{X}_4\,)^T,$$

where $\bar{X}_3 \subseteq X_3$ and $\bar{X}_4 \subseteq X_4$ satisfy

$$\mathbf{A}_{33}\bar{X}_3 + \mathbf{A}_{34}\bar{X}_4 \subseteq \bar{X}_3, \quad \mathbf{C}_3\bar{X}_3 + \mathbf{C}_4\bar{X}_4 = 0.$$

Substitution for $\bar{X}_3$ yields $(\mathbf{A}_{33} - \mathbf{A}_{34}(\mathbf{C}_4^T\mathbf{C}_4)^{-1}\mathbf{C}_4^T\mathbf{C}_3)\bar{X}_3 \subseteq \bar{X}_3$, i.e., $\bar{\mathbf{A}}_{33}\bar{X}_3 \subseteq \bar{X}_3$, and $\mathbf{C}_3\bar{X}_3 - \mathbf{C}_4(\mathbf{C}_4^T\mathbf{C}_4)^{-1}\mathbf{C}_4^T\mathbf{C}_3\bar{X}_3 = (\mathbf{I} - \mathbf{P}_o)\mathbf{C}_3\bar{X}_3 = 0$. Since $(\mathbf{I} - \mathbf{P}_o)$ is symmetric and positive semidefinite, $\bar{\mathbf{C}}_3$ can be written $\bar{\mathbf{C}}_3 := (\mathbf{I} - \mathbf{P}_o)^{1/2}\mathbf{C}_3$, so that $\bar{X}_3 \subseteq \ker\bar{\mathbf{C}}_3$. Since $\mathbf{C}_4$ has full rank, it follows that $\bar{V}_3 \neq 0 \Leftrightarrow V_3 \neq 0$ •

In conclusion, the trajectories, in the limit, of the optimal system are governed by the following equations:

$$\begin{aligned}&\dot{\hat{\mathbf{x}}}_2 = (\mathbf{A}_{22}\hat{\mathbf{x}}_2 - \mathbf{B}_2\mathbf{B}_2^T\hat{\mathbf{P}}_2)\hat{\mathbf{x}}_2 \\ &\dot{\mathbf{x}}_1 = (\mathbf{A}_{11} - \mathbf{A}_{14}\mathbf{A}_{14}^T\mathbf{P}_1 - \mathbf{A}_{14}\mathbf{A}_{34}^T\mathbf{P}_2^T)\mathbf{x}_1 + \\ &+(\mathbf{A}_{13} - \mathbf{A}_{14}\mathbf{A}_{34}^T\hat{\mathbf{P}}_4 - \mathbf{A}_{14}\mathbf{A}_{14}^T\mathbf{P}_2 - \mathbf{A}_{14}\mathbf{A}_{34}^T\bar{\mathbf{P}}_4)\mathbf{x}_3 \\ &\dot{\mathbf{x}}_3 = (-\mathbf{A}_{34}\mathbf{A}_{14}^T\mathbf{P}_1 - \mathbf{A}_{34}\mathbf{A}_{34}^T\mathbf{P}_2^T)\mathbf{x}_1 + \\ &+(\mathbf{A}_{33} - \mathbf{A}_{34}\mathbf{A}_{34}^T\hat{\mathbf{P}}_4 - \mathbf{A}_{34}\mathbf{A}_{14}^T\mathbf{P}_2 - \mathbf{A}_{34}\mathbf{A}_{34}^T\bar{\mathbf{P}}_4)\mathbf{x}_3\end{aligned}$$

Furthermore, the eigenvalues of the subspace $B/(B \cap R^*)$ go to $-\infty$ in the limit. As in the case of relative degree (1,...,1), the unstable eigenvalues of V^*/R^* go to their mirror images, i.e., are reflected in the imaginary axis, as $\epsilon \to 0$.

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